15-251: Great Theoretical Ideas In Computer Science

Homework 3 (due February 6, 2007)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof.*

1. Pizza Cutting (15 points)

After a long day of 251 homework, you and your friend decide to treat yourself to a circular 12-cut pepperoni pizza. When the pizza gets to your place, Vocelli's makes dividing the pizza hard on you, by putting a different number of pieces of pepperoni on each slice. You and your friend decide to divide the pizza in the following way. First, you choose and eat any slice from the pizza. Then, you both alternate turns by taking and eating a slice from the pizza, but only one of the slices that borders the gap left by the removed slices.

Is there a strategy you can use to ensure that you will have eaten at least as many pieces of pepperoni as your friend, once the pizza is fully consumed? Prove that your strategy will always work, no matter how the pepperoni is distributed.

2. Finkelberg's 101 Game (25 points)

A board of Finkelberg's 101 Game consists of a row of x boxes, which is initially empty. The players take turns selecting an empty box and putting either a 0 or a 1 in it. The player who first succeeds in completing 101 in consecutive boxes wins the game. If the whole board gets filled up without a 101 appearing consecutively anywhere, the game is a draw.

a) (5 points) Show that if $x = 4$ and the first player puts a 1 in the first box, the second player can win the game.

b) (5 points) Show that if $x = 7$, the first player can win the game.

c) (15 points) Show that if $x = 2000$, the second player can win the game.

Hint: Define a "deathtrap" as a box which, given the current configuration of the board, has the following property: if a player puts anything there (either a 1 or a 0), then the other player can guarantee a win in the next move.

3. Dividing and Conquering (15 points)

Set up two piles with (m, n) chips. Two players alternate turns by choosing one of the piles; all chips in that pile are thrown away, and the remaining chips are distributed among the two piles however that player wants. Whoever throws away the last chip wins.

This is a fairly simple game. Identify all (m, n) pairs that end up being P-positions in this game. In your solution, you must prove that you have identitfied any and all possible (m, n) pairs.

4. Guatemala Revisited (25 points)

This month, Luis and his company of Guatemalans were not as fortunate in their endeavors; their only net profit for the month was a Pert Plus sample they found in the junk mail. However, they decide to divide up their earnings the same way as last time.

First, the fifty (50) Guatemalans order themselves from the least handsome to the most handsome, with Luis at the front. Starting from the most handsome, the Guatemalan at the front of the line decides who to give the sample to, and the proposal is voted on. If at least 50% of the remaining Guatemalans accept the proposal, then the chosen person will get the sample. If the proposal is rejected, Luis will be made to do janitorial duty, and the next handsomest offers a different proposal, with the same rules as before. If his proposal is rejected, then he will join Luis in janitorial duty, and the third handsomest will choose the distribution, and so on. Those doing janitorial duty cannot vote.

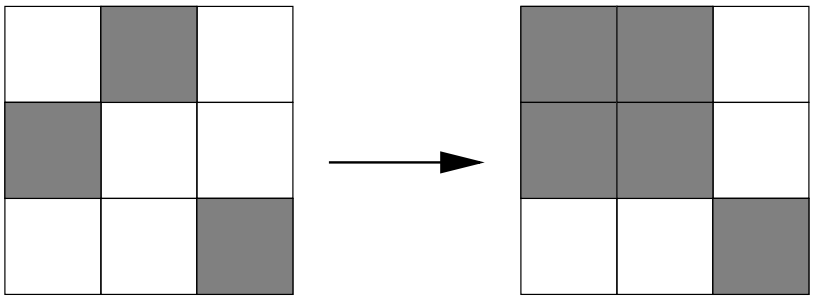
(A) These Guatemalans prefer not to do janitorial duty over everything else. (B) If they will not be doing janitorial duty, they would prefer to have the Pert Plus sample. Given (A) and (B), if the Guatemalans are still indifferent between two choices, they'd always rather vote to reject a proposal than to accept it. You may assume that everyone in the company is perfectly logical.

How many Guatemalans end up doing janitorial duty? Prove that your answer is correct.

5. Fungus Amongus (20 points)

An important concept that will help you in problem solving is the concept of an **invariant**. An invariant is a variable that does not change, no matter how the rest of the environment transforms. For example, an invariant in a poker game is the total amount of money in play; no matter how money changes hands, the sum of every player's total remains the same. Identifying an invariant is probably the best and simplest way to solve the following problem:

As part of a science experiment, a strange fungus is distributed across a $n \times n$ box of 1×1 squares. Any number of squares in the box are labelled as "starting squares", and are coated with the fungus and left to grow. This fungus reproduces in the following way: if any empty square neighbors at least two squares with fungus, then that square will start growing the fungus after one hour (neighbors are any two squares that share a border.) As an example, consider the following figure, where dark squares represent the growth of the fungus over one hour.



The goal of the science experiment is to have the entire $n \times n$ box covered with the fungus. This process can take as long as you need, as long as it will be covered eventually.

a) (5 points) Prove that there exists a way to choose n starting squares that will eventually cause the entire box to be covered.

b) (15 points) Is there a way to cover the entire box with fungus if you start with fewer than n starting squares? Prove that your answer is correct.

6. Puzzlehunt (Extra Credit, 10 points)

The easiest extra credit problem of the year! Give us ideas for the next puzzlehunt: potential puzzles, how to make the scavenger hunt more fun, or other feedback on how to improve. **Note:** Only ideas that we like will get the ten (10) points, so be sure to think these through before you submit.